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Exponential Synchronization of Quaternion-Valued Inertial Neural Networks with Mixed Delays

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ABSTRACT
This paper investigates the global exponential synchronization of a class of quaternion-valued inertial neural networks with mixed delays. First, a neural network model incorporating both discrete and distributed delays is established. On this basis, to overcome the limitations of traditional methods, a novel quaternion-valued controller is designed without the need for model reduction or quaternion decomposition, aiming to achieve global exponential synchronization of the system. Furthermore, by constructing an appropriate Lyapunov function and combining the algebraic properties of quaternions with relevant inequality techniques, sufficient criteria for achieving synchronization are rigorously derived within the framework of Lyapunov stability theory. Finally, numerical simulations are conducted to verify the effectiveness of the proposed control strategy and to validate the theoretical results.

KEYWORDS
Inertial neural networks; Exponential synchronization; Quaternion-valued; Mixed time delays

1 Introduction

Neural networks have been widely applied in fields such as image classification and face recognition. To enhance their capability in processing dynamic data, inertial neural networks (INNs) have been developed by introducing an inertial term to simulate the memory characteristics of biological neurons, thereby effectively improving convergence speed and robustness. Building on this foundation, quaternion-valued inertial neural networks (QINNs) further integrate the structured modeling capability of quaternion algebra with the dynamic memory advantages of inertial terms, exhibiting exceptional anti-interference performance and fault tolerance in tasks such as aerospace attitude control and three-dimensional vision. Consequently, QINNs have emerged as an effective framework for processing multimodal high-dimensional dynamic data[1-2].

However, time delays, as an inherent phenomenon in signal transmission, can easily induce oscillations and even instability in systems. The mixed time-delays model, which integrates discrete, proportional, and distributed delays, constructs a dynamic system that more closely reflects reality and is of critical value in enhancing robustness. Synchronization, as an important dynamical behavior of neural networks, has attracted considerable attention due to its applications in fields such as secure communication and cooperative control. However, current research on inertial neural networks with mixed delays is still mainly concentrated in the real-valued or complex-valued domain, while related studies within the quaternion framework are still in their infancy, leaving a significant theoretical gap that urgently needs to be filled[3-4].

To address this gap, this paper investigates the exponential synchronization of quaternion-valued inertial neural networks with mixed delays by employing a combined non-reduction and non-separation method. This approach not only fully preserves the inertial characteristics of the system but also avoids the traditional approach of decomposing the quaternion-valued system into two complex-valued or four real-valued subsystems, thereby effectively reducing the complexity of theoretical derivation. On this basis, sufficient criteria for achieving exponential synchronization are established, and the validity of the theoretical results is further verified through numerical simulations. This study provides theoretical support for the application of quaternion-valued inertial neural networks in related fields.

2 Preliminaries

Consider the following class of QIDNNS with time delays:

$$\ddot{x}_p(t)=-a_p\dot{x}_p(t)-b_p x_p(t)+\sum_{q=1}^{n}c_{pq}f_q(x_q(t))+\sum_{q=1}^{n}d_{pq}f_q(x_q(t-\tau(t)))+\sum_{q=1}^{n}m_{pq}\int_{t-v(t)}^{t}f_q(x_q(\eta))d\eta \tag{1}$$

where $p=1,2\cdots n, x_p\in Q$ denotes the state of the p-th neuron at time t; $a_p>0$ and $b_p>0$ are system parameters; $c_{pq}, d_{pq}, m_{pq}\in Q$ are the connection weights, discrete delay weights, and distributed delay weights, respectively; $f_q(\cdot)\in Q$,

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are the activation functions; $\tau(t)$ and $\upsilon(t)$ is the discrete time delays, and $0 \le \tau(t) \le \tau$, $\dot{\tau}(t) \le \tilde{\tau} < 1$, $0 \le \upsilon(t) \le \breve{\upsilon}$, $\dot{\upsilon}(t) \le \tilde{\upsilon} < 1$.

The initial conditions for system (1) are given by:

$$x_p(\varsigma) = \varphi_p(\varsigma),\ \dot{x}_p(\varsigma) = \psi_p(\varsigma),\ \varsigma \in [-\tau, 0] \tag{2}$$

In which $\varphi_p(\varsigma)$ and $\psi_p(\varsigma)$ are continuous.

If system (1) is regarded as the drive system, then the corresponding response system is given by:

$$\ddot{y}_p(t) = -a_p \dot{y}_p(t) - b_p y_p(t) + \sum_{q=1}^{n} c_{pq} f_q(y_q(t)) + \sum_{q=1}^{n} d_{pq} f_q(y_q(t-\tau(t))) + \sum_{q=1}^{n} m_{pq} \int_{t-\upsilon(t)}^{t} f_q(y_q(\eta)) d\eta + U_p(t) \tag{3}$$

here $y_p(t)$denotes the state of the p-th neuron in the response system, $U_p(t)$represrnts a control input.

The initial conditions for system (3) are given by:

$$y_p(\varsigma) = \overline{\varphi}_p(\varsigma),\ \dot{y}_p(\varsigma) = \overline{\psi}_p(\varsigma),\ \varsigma \in [-\tau, 0] \tag{4}$$

In which$\overline{\varphi}_p(\varsigma)$and $\overline{\psi}_p(\varsigma)$ are continuous.

The synchronization error is defined as $w_p(t) = y_p(t) - x_p(t)$. Then, the error system can be described as follows:

$$\ddot{w}_p(t) = -a_p \dot{w}_p(t) - b_p w_p(t) + \sum_{q=1}^{n} c_{pq} \tilde{f}_q(w_q(t)) + \sum_{q=1}^{n} d_{pq} \tilde{f}_q(w_q(t-\tau(t))) + \sum_{q=1}^{n} m_{pq} \int_{t-\upsilon(t)}^{t} \tilde{f}_q(w_q(\eta)) d\eta + U_p(t) \tag{5}$$

where the feedback controller is designed as

$$U_p(t) = -\eta_p w_p(t) - \xi_p w_p(t) \tag{6}$$

with $\eta_p > 0$ and $\xi_p > 0$ being the control gains.

Assumption 1. There exist positive numbers L_r,such that $|f_p(\iota) - f_p(r)| \le L_r |\iota - r|$, $\iota, r \in Q$.

Definition1. For any two solutions $x(t) \in Q$ and $y(t) \in Q$, if there exist a constant $\varepsilon > 0$and $\tilde{M} \ge 1$such that $\|w(t)\| \le \tilde{M}(sup\|w(t)\| + \|\dot{w}(t)\|)e^{-\varepsilon t}$, then the quaternion-valued inertial neural systems (1) and (2) are said to be exponentially synchronous under the feedback controller.

3 Main Conclusions

In this section, the exponential synchronization criteria for QINNS are presented using the non-separation method.

Theorem 1. Based on the Assumptions1 , there exist positive numbersλ_p, μ_p, *and* ε such that $B_p = \begin{pmatrix} \Xi_r & \Lambda_r \\ \Lambda_r & \Omega_r \end{pmatrix} \le 0$, in which

$$\Xi_r = \lambda_p \epsilon + \mu_p \epsilon - \mu_p \xi_p - \mu_p b_p^R + \mu_p (|b_p^R| + |b_p^J| + |b_p^K|) + \mu_p (|a_p^R| + |a_p^J| + |a_p^K|) + \sum_{q=1}^{n} \mu_q L_p |c_{qp}|$$

$$+ \frac{1}{2} \sum_{q=1}^{n} \mu_p L_q \left(|c_{qp}| + |d_{pq}| + \nu |m_{pq}| \right) + \sum_{q=1}^{n} \frac{\mu_p L_q |d_{pq}|}{1-\tilde{\tau}} e^{2\varepsilon\tau} + \sum_{q=1}^{n} \frac{\mu_p L_p |m_{pq}| \upsilon e^{2\varepsilon\upsilon}}{1-\tilde{\upsilon}},$$

$$\Lambda_r = \frac{1}{2}\lambda_p + \mu_p \varepsilon + \frac{1}{2}\mu_p - \frac{1}{2}\mu_p \eta_p - \frac{1}{2}\mu_p \xi_p - \mu_p a_p^R - \mu_p b_p^R,$$

$$\Omega_r = \mu_p \epsilon + \mu_p - \mu_p \eta_p - \mu_p a_p^R + \mu_p (|b_p^R| + |b_p^J| + |b_p^K|) + \mu_p (|a_p^R| + |a_p^J| + |a_p^K|$$

$$+ \frac{1}{2} \sum_{q=1}^{n} \mu_p L_q \left(|c_{qp}| + |d_{pq}| + \nu |m_{pq}| \right),$$

then systems (1) and (2) achieve exponential synchronization under controller (6).

Proof: Construct the following Lyapunov functional:

$$V(t) = \frac{1}{2} \sum_{p=1}^{n} \lambda_p e^{2\varepsilon t} w_p(t) \overline{w_p(t)} + \frac{1}{2} \sum_{p=1}^{n} \mu_p e^{2\varepsilon t} (w_p(t) + \dot{w}_p(t)) \overline{(w_p(t) + \dot{w}_p(t))} + \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\mu_p L_p |d_{pq}|}{1-\tilde{\tau}} e^{2\varepsilon\tau}$$

$$* \int_{t-\tau(t)}^{t} e^{2\varepsilon h} w_q(h) \overline{w_q(h)} dh + \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\mu_p L_p |m_{pq}| e^{2\varepsilon\upsilon}}{1-\tilde{\nu}} \int_{-\upsilon(t)}^{0} \int_{t+l}^{t} w_q(\eta) \overline{w_q(\eta)} e^{2\varepsilon\eta} d\eta dl \tag{7}$$

By computing the derivative of $V(t)$ and by Assumption 1 and 2,

$$\dot{V}(t) \le \sum_{p=1}^{n} \lambda_p \varepsilon e^{2\varepsilon t} w_p(t) \overline{w_p(t)} + \frac{1}{2} \sum_{p=1}^{n} \lambda_p e^{2\varepsilon t} (\dot{w}_p(t) \overline{w_p(t)} + w_p(t) \overline{\dot{w}_p(t)}) + \sum_{p=1}^{n} \mu_p \varepsilon e^{2\varepsilon t} (w_p(t) + \dot{w}_p(t)) \overline{(w_p(t) + \dot{w}_p(t))}$$

$$+ \frac{1}{2} \sum_{p=1}^{n} \mu_p e^{2\varepsilon t} (\dot{w}_p(t) + \ddot{w}_p(t)) \overline{(w_p(t) + \dot{w}_p(t))} + \frac{1}{2} \sum_{p=1}^{n} \mu_p e^{2\varepsilon t} (w_p(t) + \dot{w}_p(t)) \overline{(\dot{w}_p(t) + \ddot{w}_p(t))}$$

$$+ e^{2\varepsilon t} \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\mu_p L_p |d_{pq}|}{1-\tilde{\tau}} e^{2\varepsilon\tau} w_q(t) \overline{w_q(t)} - e^{2\varepsilon t} \sum_{p=1}^{n} \sum_{q=1}^{n} \mu_p L_p |d_{pq}| e^{2\varepsilon\tau} w_p(t-\tau(t)) \overline{w_p(t-\tau(t))}$$

$$
\begin{aligned}
&+e^{2\varepsilon t}\sum_{p=1}^{n}\sum_{q=1}^{n}\frac{\mu_p L_p|m_{pq}|\upsilon e^{2\varepsilon\upsilon}}{1-\tilde{\upsilon}}w_q(t)\,\overline{w_q(t)}-e^{2\varepsilon t}\sum_{p=1}^{n}\sum_{q=1}^{n}\mu_p L_p|m_{pq}|\int_{t-\upsilon(t)}^{t}w_q(\eta)\,\overline{w_q(\eta)}\,e^{2\varepsilon\eta}d\eta\\
&\le e^{2\varepsilon t}\sum_{p=1}^{n}\Bigg(\lambda_p\epsilon+\mu_p\epsilon-\mu_p\xi_p-\mu_p b_p^R+\mu_p(|b_p^R|+|b_p^J|+|b_p^K|)+\mu_p(|a_p^R|+|a_p^J|+|a_p^K|)+\sum_{q=1}^{n}\mu_q L_p|c_{qp}|\\
&\quad+\frac{1}{2}\sum_{q=1}^{n}\mu_p L_q\left(|c_{qp}|+|d_{pq}|+\nu|m_{pq}|\right)+\sum_{q=1}^{n}\frac{\mu_p L_q|d_{pq}|}{1-\tilde{\tau}}e^{2\varepsilon\tau}+\sum_{q=1}^{n}\frac{\mu_p L_p|m_{pq}|\upsilon e^{2\varepsilon\upsilon}}{1-\tilde{\upsilon}}\Bigg)w_p(t)\,\overline{w_p(t)}\\
&+e^{2\varepsilon t}\sum_{p=1}^{n}\left(\frac{1}{2}\lambda_p+\mu_p\varepsilon+\frac{1}{2}\mu_p-\frac{1}{2}\mu_p\eta_p-\frac{1}{2}\mu_p\xi_p-\mu_p a_p^R-\mu_p b_p^R\right)w_p(t)\,\overline{\dot{w}_p(t)}\\
&+e^{2\varepsilon t}\sum_{p=1}^{n}\left(\frac{1}{2}\lambda_p+\mu_p\varepsilon+\frac{1}{2}\mu_p-\frac{1}{2}\mu_p\eta_p-\frac{1}{2}\mu_p\xi_p-\mu_p a_p^R-\mu_p b_p^R\right)\dot{w}_p(t)\,\overline{w_p(t)}\\
&+e^{2\varepsilon t}\sum_{p=1}^{n}\Bigg(\mu_p\epsilon+\mu_p-\mu_p\eta_p-\mu_p a_p^R+\mu_p(|b_p^R|+|b_p^J|+|b_p^K|)+\mu_p(|a_p^R|+|a_p^J|+|a_p^K|)\\
&\quad+\frac{1}{2}\sum_{q=1}^{n}\mu_p L_q\left(|c_{qp}|+|d_{pq}|+\nu|m_{pq}|\right)\Bigg)\dot{w}_p(t)\,\overline{\dot{w}_p(t)}\\
&=e^{2\varepsilon t}\sum_{p=1}^{n}\left(w_p(t),\dot{w}_p(t)\right)B_p\left(w_p(t),\dot{w}_p(t)\right)^T\\
&\le 0.
\end{aligned}
\tag{8}
$$

Which reveals that $V(t) \le V(0)$. Furthermore,according to the construction of $V(t)$,

$$
\|w(t)\|\le\sqrt{\frac{2V(0)}{\xi}}\,e^{-2\varepsilon t}\le\tilde{M}\left(\sup\|w(s)\|+\sup\|\dot{w}(s)\|\right)e^{-2\varepsilon t}.
$$

then systems (1) and (3) are exponential synchronization.

4 Numerical Simulations

The following example demonstrates the effectiveness of the theoretical results.

$$
\ddot{x}_p(t)=-a_p\dot{x}_p(t)-b_p x_p(t)+\sum_{q=1}^{2}c_{pq}f_q(x_q(t))+\sum_{q=1}^{2}d_{pq}f_q(x_q(t-\tau(t)))+\sum_{q=1}^{2}m_{pq}\int_{t-v(t)}^{t}f_q(x_q(\eta))d\eta \tag{9}
$$

$$
\ddot{y}_p(t)=-a_p\dot{y}_p(t)-b_p y_p(t)+\sum_{q=1}^{2}c_{pq}f_q(y_q(t))+\sum_{q=1}^{2}d_{pq}f_q(y_q(t-\tau(t)))+\sum_{q=1}^{2}m_{pq}\int_{t-v(t)}^{t}f_q(y_q(\eta))d\eta+U_p(t) \tag{10}
$$

The activation function is defined as $f_q = tanh(x^R)+itanh(x^I)+Jtanh(x^J)+ktanh(x^K)$ for $p, q = 1,2$, the time-varying delay is given by $\tau(t)=\frac{e^t}{1+e^t}$, and the distributed delay kernel follows an exponentially decaying form. The chaotic attractor phase portraits of system (9) are displayed in Figure 1.

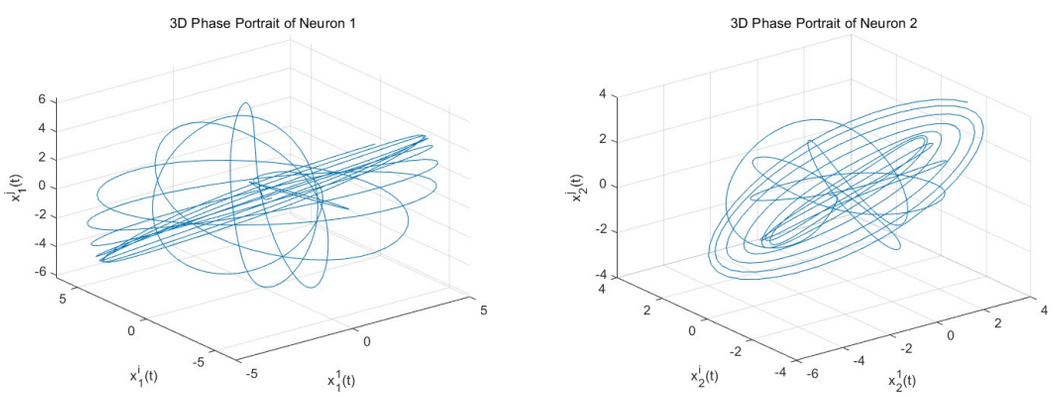

Figure 1 Chaotic Attractor of Neuron 1/2 in the R-I-J Subspace

Exponential synchronization of the master – slave systems (9) and (10) is achieved using the aperiodic intermittent control strategy given in (6). The synchronization results are illustrated in Figure 2..

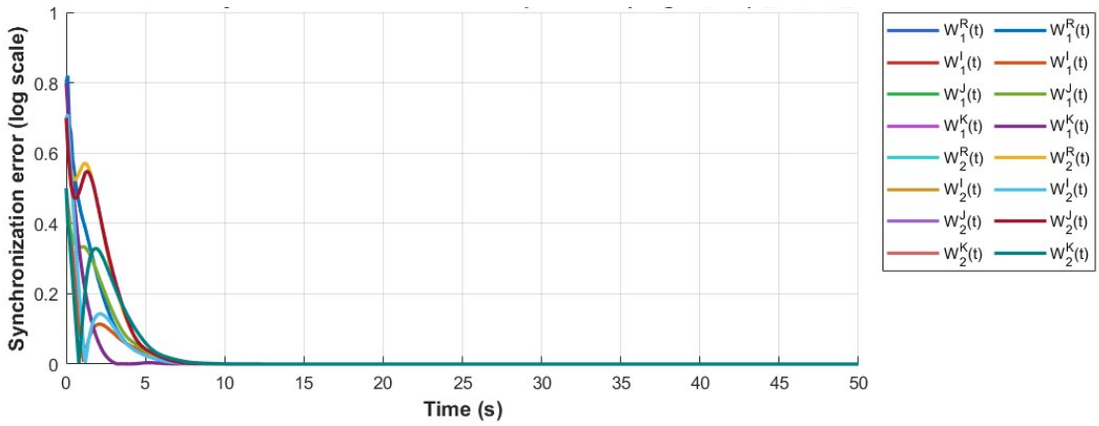

Figure 2 Evolution of Synchronization Error Components

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